III. Effective Interaction Theory

Topics to be covered include:
- Intuitive ideas
- Brief review of operator formalism
- Watson scattering formalism
- Kerman, McManus, Thaler scattering formalism
- Feshbach scattering formalism
- Brueckner nuclear matter formalism
- Relativistic multiple scattering

General References:
The intuitive ideas that led to effective interaction theory

The basic ideas for dealing with the many-body, strong (non-perturbative) nuclear interaction problem began with scattering – so that’s where I will start. The seminal idea was due to Leslie L. Foldy who as a recent Physics B.S. graduate (in 1941) was working on sonar during WWII in New York City. Building on his experiences with acoustical waves Foldy in 1945, just before entering graduate school to work with J. R. Oppenheimer at U.C. Berkeley, and after his work was declassified, published the landmark paper (L. L. Foldy, Phys. Rev. 67, 107 (1945)).

Foldy described projectile scattering from a nucleus as a wave propagating through many, dense scattering sources with a complex (absorptive) index of refraction. His essential idea was to express the total scattered wave in terms of individual N+N scattered waves, rather than in terms of the very strong N+N interaction which cannot be expanded in a perturbation series, and may even diverge in the case of hard-core N-N interactions. A nice tribute to Foldy is:


“His 1945 theory of the multiple scattering of waves laid out the fundamentals that most modern theories have followed (and sometimes rediscovered),…”

Not bad for a pre-graduate student!
The intuitive ideas that led to effective interaction theory

In 1950 Geoffrey Chew introduced the “impulse approximation” as a suitable way to simplify the intractable A+1 – body problem (e.g. p + A, n + A) to an effective two-body scattering problem. Three papers established the basic ideas for what would become known as Multiple Scattering Theory:


The basics are:

1) the full A+1 scattering can be accurately represented as a coherent sum of individual hadron + nucleon scatterings and re-scatterings from nucleons in the target nucleus
2) at high energies the free-space hadron + nucleon scattering amplitude is unaffected by the nuclear medium
3) the hadron + nucleus scattering amplitude should be expanded in terms of the two-body scattering amplitudes, rather than directly in terms of the N+N potential.

In 1951 Melvin Lax (Rev. Mod. Phys. 23, 287 (1951), Phys. Rev. 85, 621 (1952)) extended these approaches to obtain an effective interaction potential, later called the optical potential to represent the effective p + A interaction. This was the first representation of such an effective potential and introduced the so-called “tp” form, where $\rho$ is the nuclear density and $t$ is an effective N+N interaction. Hans Bethe learned about this and in a seminar passed this new idea on to Roy Thaler (my 2nd mentor in the late 70’s and 80’s) who devoted his career to multiple scattering formalism.
In 1953 **Kenneth M. Watson** gathered up all these emerging ideas and published the first, formal scattering solution for the p + A problem in *K. M. Watson, Phys. Rev. 89, 575 (1953)*. His theory will be presented in later slides.

In 1959 **Arthur Kerman, Hugh McManus** and **Roy Thaler** corrected a double counting problem in the Watson theory by re-organizing the expansions (Ann. Phys. 8, 551 (1959)) which paved the way for accurate application of Chew’s impulse approximation and led to many applications for scattering experiments.
The intuitive ideas that led to effective interaction theory

Herman Feshbach and collaborators, at the same time, developed a powerful projection operator formalism which they used to generate a perturbation expansion of the optical potential and which could be applied to reactions other than elastic scattering [Ann. Rev. Nucl. Sci 8, 49 (1958); Ann. Phys. (NY) 5, 357 (1958); Ann. Phys. (NY) 19, 287 (1962)].

The success of multiple scattering formalism and the effective interactions was noticed by nuclear structure theorists. These ideas were incorporated into theories of infinite nuclear matter by Keith Brueckner and Levinson in 1955 (Phys. Rev. 97, 1344 (1955)). Infinite nuclear matter (no Coulomb, no surface, no symmetry energies to worry about) was a first step on the way to a theory of nuclear structure. The resulting effective interactions, called “g-matrix” in the literature, is ubiquitous in nuclear structure calculations.
Introduction to Operator Formalism

I assume that most of this notation is familiar to you and here offer only a quick review of those expression which are most useful in nuclear scattering and which we will need to understand the effective interaction formalism. The following are copied from Rodberg and Thaler, Ch. 6.

Thus, the projection of the vector $\psi$ on the eigenvector $\phi_a$ is just the scalar product, as expected. If we insert this result into the expansion, we have

$$|\psi\rangle = \sum_a |\phi_a\rangle \langle \phi_a | \psi\rangle. \quad (1.16)$$

Thus a mathematical statement of the completeness of the set $\phi_a$, that is, of its ability to provide an expansion of any state, is the relation

$$1 = \sum_a |\phi_a\rangle \langle \phi_a |. \quad (1.17)$$

This complicated way of writing the unit operator is useful in a great many problems. It provides a straightforward means of expressing $\psi$ in any desired representation.
Introduction to Operator Formalism

Two particularly useful representations are the spatial or coordinate representation and the momentum representation. In the coordinate representation eigenstates of position, denoted by $| \phi_\mathbf{r} \rangle$ or $| \mathbf{r} \rangle$, are used. Here $\mathbf{r}$ may be the position of a single particle, or it may be the ensemble of the position vectors of many particles. Since $\mathbf{r}$ is a continuous variable, these states are normalized according to

$$
| \mathbf{r} \rangle | r' \rangle = \delta(\mathbf{r} - \mathbf{r}'), \quad (1.18)
$$

and the completeness statement is

$$
\int | \mathbf{r} \rangle d^3r \langle \mathbf{r} | = 1. \quad (1.19)
$$

In this representation an arbitrary state vector $\psi$ is written

$$
| \psi \rangle = \int | \mathbf{r} \rangle d^3r \langle \mathbf{r} | \psi \rangle. \quad (1.20)
$$

This has a simple interpretation in terms of our previous work. The ket-vector $| \mathbf{r} \rangle$ represents a system localized at $\mathbf{r}$. The scalar product $\langle \mathbf{r} | \psi \rangle$ is then the amplitude for finding the system at $\mathbf{r}$, which is the Schrödinger wave function. Thus we have

$$
\langle \mathbf{r} | \psi \rangle \equiv \psi(\mathbf{r}), \quad (1.21)
$$

in our previous notation.

Momentum eigenstates are denoted by $| k \rangle$. Conventionally, they are normalized so that

$$
\langle k | k' \rangle = (2\pi)^3 \delta(\mathbf{k} - \mathbf{k'}). \quad (1.22)
$$

The completeness relation is then

$$
(2\pi)^{-3} \int | k \rangle d^3k \langle k | = 1, \quad (1.23)
$$
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and the momentum expansion of the state vector is

$$|\psi\rangle = (2\pi)^{-3} \int |k\rangle \, d^3k \, \langle k | \psi \rangle. \quad (1.24)$$

The relation between these two representations is easy to see. Let us suppose $\psi$ to be a momentum eigenstate $|k\rangle$; then $\langle r | k \rangle$ is the wave function for a system having a wave number $k$, or explicitly

$$\langle r | k \rangle = A_k \, e^{ik \cdot r}. \quad (1.25)$$

The normalization constant $A_k$ is obtainable from the normalizations of the vectors $|r\rangle$ and $|k\rangle$. If we use the completeness relation for position eigenstates and the normalization condition for the momentum eigenstates, we have

$$(2\pi)^3 \, \delta(k - k') = \langle k | k' \rangle = \int \langle k | r \rangle \, d^3r \, \langle r | k' \rangle$$

$$= A_k^* A_k \, \int d^3r \, \exp[i(k' - k) \cdot r]. \quad (1.26)$$

The last integral in Eq. (1.26) is $(2\pi)^3 \, \delta(k - k')$, so that $A_k$ can be chosen to have a phase such that $A_k = 1$, from which it follows that

$$\langle r | k \rangle = e^{ik \cdot r}. \quad (1.27)$$

If we insert the completeness relation for momentum eigenstates into the coordinate representation of a general state, we find

$$\langle r | \psi \rangle = (2\pi)^{-3} \int \langle r | k \rangle \, d^3k \, \langle k | \psi \rangle$$

$$= (2\pi)^{-3} \int d^3k \, e^{ik \cdot r} \, \langle k | \psi \rangle. \quad (1.28)$$

Hence the function $\langle r | \psi \rangle$, the coordinate representative of $\psi$, is just the Fourier transform of the function $\langle k | \psi \rangle$, the momentum representative.
Introduction to Operator Formalism

2. Operator Form of the Scattering Equations

The central results of the earlier chapters may be summarized in the following equations:

(a) Differential equation for the wave function:

\[ \{ E + (\hbar^2/2m)V^2 \} \psi(r) = V(r) \psi(r); \]  \hspace{1cm} (2.1)

(b) Integral equation for the wave function:

\[ \psi_{k}^{(+)}(r) = \phi_{k}(r) + \int d^3r' \, G^{(+)}(r, r') \, V(r') \, \psi_{k}^{(+)}(r'), \]  \hspace{1cm} (2.2)

with

\[ G^{(+)}(r, r') = -\left( \frac{2m}{4\pi\hbar^2} \right) \frac{\exp(ik|r - r'|)}{|r - r'|} \]  \hspace{1cm} (2.3)

and

\[ \phi_{k}(r) = e^{ik \cdot r}; \]  \hspace{1cm} (2.4)

(c) Integral expression for the scattering amplitude:

\[ f(\theta) = -\left( \frac{2m}{4\pi\hbar^2} \right) \int d^3r \, \phi_{k}^{+}(r) \, V(r) \, \psi_{k}^{(+)}(r); \]  \hspace{1cm} (2.5)

(d) Integral equation for the transition amplitude:

\[ T(k', k) = V(k', k) + (2m/\hbar^2)(2\pi)^{-3} \int d^3k'' \, \frac{V(k', k'') \, T(k'', k)}{k^2 - k''^2 + i\epsilon}, \]  \hspace{1cm} (2.6)

with

\[ V(k', k) = \int d^3r \, e^{-ik \cdot r} \, V(r) \, e^{ik \cdot r}. \]  \hspace{1cm} (2.7)

We now express these equations in operator language.
Introduction to Operator Formalism

The Schrödinger Equation

The wave function $\psi(r)$ is the spatial representative of a state vector $\psi$, and we write

$$\psi(r) = \langle r | \psi \rangle.$$  

(2.8)

The potential $V(r)$ is related to the spatial representative of a potential operator $V$. The state vector $V\psi$ has the spatial representative

$$\langle r | V | \psi \rangle = \int \langle r | V | r' \rangle d^3r' \langle r' | \psi \rangle$$

$$= \int d^3r' \langle r | V | r' \rangle \psi(r').$$  

(2.9)

Since we want the operator $V$ to reproduce the potential energy term in the Schrödinger equation, Eq. (2.1), we must require

$$\int d^3r' \langle r | V | r' \rangle \psi(r') = V(r) \psi(r).$$  

(2.10)

Thus the spatial representative of the potential operator $V$ is

$$\langle r | V | r' \rangle = \delta(r - r') V(r).$$  

(2.11)  

Local Potential

energy term of the Schrödinger equation.

We can now write the Schrödinger equation as

$$E \langle r | \psi \rangle - \langle r | H_0 | \psi \rangle = \langle r | V | \psi \rangle.$$  

(2.15)

This is just the spatial representative of the operator equation

$$(E - H_0)\psi = V\psi$$  

(2.16)  

Schroedinger equation in operator form

or

$$(E - H)\psi = 0.$$  

(2.17)
Introduction to Operator Formalism

**Integral Equation for the Wave Function**

The integral equation for $\psi_k^{(+)}(r)$ contains two new elements. One is very simple and is just the coordinate representative of a state vector $\phi_k = |k\rangle$ for the initial unperturbed state:

$$\phi_k(r) = \langle r | \phi_k \rangle = \langle r | k \rangle = e^{ik \cdot r}. \tag{2.18}$$

The second element, the Green’s function for the unperturbed Hamiltonian $H_0$, is more interesting.

We recall that the Green’s function $G(r, r')$ describes the propagation of a particle from $r'$ to $r$. It is thus natural to represent it by an operator $G$ which has the effect of transforming a state localized at $r'$ into one localized at $r$. This is a nonlocal operator, and its spatial representative is written

$$\langle r | G | r' \rangle = G(r, r'). \tag{2.19}$$

We now want to discuss a way of expressing the operator $G$ in terms of a Hamiltonian $H_0$. The differential equation for the Green’s function is

$$\{E + (\hbar^2/2m) \nabla^2\} G(r, r') = \delta(r - r'). \tag{2.20}$$

If we note that the delta function is the coordinate representative of the unit operator and use the definition of $H_0$ given by Eq. (2.14), we see that this is equivalent to the operator equation

$$(E - H_0)G = 1. \tag{2.21}$$

Thus formally the operator $G$ is

$$G = (E - H_0)^{-1} = 1/(E - H_0), \tag{2.22}$$

provided that the inverse operator exists.

To examine whether this operator exists, let us consider $G$ in the momentum representation. The states $|k\rangle$ are eigenstates of $H_0$ having energy
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\[ E' = \frac{\hbar^2 k'^2}{2m}, \text{ so that } \]
\[ (E' - H_0) |k'\rangle = 0. \tag{2.23} \]

In this representation \( G \) is

\[ \langle k'' | G | k' \rangle = \langle k'' | \frac{1}{E - H_0} | k' \rangle = (2\pi)^3 \delta(k'' - k) \frac{1}{E - E'}. \tag{2.24} \]

Since the eigenvalues \( E' \) span all real positive numbers, the energy denominator can vanish when \( E \) is real. The right-hand side of Eq. (2.24) is therefore singular and is undefined when \( E = E' \).

To more closely define the inverse operator, we must specify the behavior of the inverse matrix in the neighborhood of the singularity. This can be done if we use the Fourier expansion of the Green's function discussed in Chapter 4. There we found that the outgoing-wave Green's function can be written

\[ G^{(+)}(r, r') = (2m\hbar^2)(2\pi)^{-3} \int d^3 k' \frac{\exp[i k' \cdot (r - r')]}{k^2 - k'^2 + i\epsilon}. \tag{2.25} \]

In this and all subsequent formulas it is understood that the limit \( \epsilon \to 0 \) is to be taken after the integral is performed. Written in the Dirac notation, this integral becomes

\[ G^{(+)}(r, r') = (2m\hbar^2)(2\pi)^{-3} \int d^3 k' \frac{\langle r | k' \rangle \langle k' | r' \rangle}{k^2 - k'^2 + i\epsilon}. \tag{2.26} \]

or, equivalently,

\[ G^{(+)}(r, r') = (2m\hbar^2)(2\pi)^{-3} \int d^3 k' \]
\[ \times \frac{\langle r | k' \rangle \delta(k' - k'') d^3 k'' \langle k'' | r' \rangle}{k^2 - k''^2 + i\epsilon}. \tag{2.27} \]
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We now recognize that, as in Eq. (2.24), we can express the denominator as the matrix element of an operator:

\[
\left(\frac{2m}{\hbar^2}\right) \frac{(2\pi)^3 \delta(k' - k'')} {k^2 - k''^2 + i\epsilon} = \langle k' \mid \frac{1} {E - H_0 + i\epsilon} \mid k''\rangle.
\]  

(2.28)

Then the Green's function is

\[
G^{(+)}(r, r') = (2\pi)^{-6} \int \langle r \mid k'\rangle d^3k' \langle k' \mid \frac{1} {E - H_0 + i\epsilon} \mid k''\rangle d^3k'' \langle k'' \mid r'\rangle
\]

\[
= \langle r \mid \frac{1} {E - H_0 + i\epsilon} \mid r'\rangle.
\]  

(2.29)

Our final well-defined result for the operator form of the outgoing-wave Green's function is therefore

\[
G^{(+)} = \frac{1} {E - H_0 + i\epsilon}.
\]  

(2.30)

The inverse operator appearing here or in Eq. (2.30) can be written formally as

\[
P\left(\frac{f(x)} {x}\right) = \lim_{\epsilon \to 0} \left[ \int_{-\infty}^{-\epsilon} \frac{f(x)} {x} dx + \int_{\epsilon}^{\infty} \frac{f(x)} {x} dx \right]
\]

\[
\frac{1} {E - H_0 + i\epsilon} = \mathcal{P} \frac{1} {E - H_0} - i\pi \delta(E - H_0),
\]  

(2.38)

where \(\mathcal{P}\) denotes the Cauchy principal value. The momentum-space matrix element of this is identical with the result obtained in Chapter 4, Eq. (3.106), and illustrated in Fig. 4.5.

We now have an operator for the Green's function and can write the state vector \(\psi^{(+)}_k\) in abstract operator language as

\[
\psi^{(+)}_k = \phi_k + \frac{1} {E - H_0 + i\epsilon} V\psi^{(+)}_k.
\]  

(2.39)
Introduction to Operator Formalism

**The Wave Matrix**

In many applications the algebraic manipulations are simplified and made more transparent by the introduction of an operator to represent the effects of the scattering process upon the state vector. A scattering experiment begins with the initial state \( \phi_i \), which then develops under the influence of the scattering potential \( V \). We would thus like an operator that acts upon the state \( \phi_i \) to create the scattering state \( \psi_i^{(+)} \).

Let us then define a wave operator or wave matrix \( \Omega^{(+)} \) so that

\[
\psi_i^{(+)} = \Omega^{(+)} \phi_i.
\] (2.52)

The wave operator \( \Omega^{(+)} \) transforms a free-particle state \( \phi_i \) into a scattering state \( \psi_i^{(+)} \). In the spatial representation it is a nonlocal function \( \langle r | \Omega^{(+)} | r' \rangle \) which produces the wave function \( \psi_i^{(+)}(r) \) by the integral operation

\[
\psi_i^{(+)}(r) = \int d^3r' \langle r | \Omega^{(+)} | r' \rangle \phi_i(r').
\] (2.53)

It describes the change in the relative two-particle wave function due to the mutual interaction \( V(r) \).

The wave matrix \( \Omega^{(+)} \) satisfies the operator equation

\[
\Omega^{(+)} = 1 + \frac{1}{E - H_0 + i\varepsilon} V\Omega^{(+)},
\] (2.54)

as we see immediately from Eq. (2.39). From Eq. (2.42) an explicit formula for \( \Omega^{(+)} \) is

\[
\Omega^{(+)} = 1 + \frac{1}{E - H + i\varepsilon} V.
\] (2.55)
THE SCATTERING AMPLITUDE AND THE $T$ MATRIX

The previous results enable us immediately to write the scattering amplitude in operator language. If the initial state is $\phi_i$ and the final state is $\phi_j$, we have

$$f(\theta) = -(2m/4\pi \hbar^2) \langle \phi_j \mid V \mid \psi_i^{(+)} \rangle.$$  \hfill (2.63)

The transition amplitude is just

$$T(k', k) = -(4\pi \hbar^2 /2m) f(\theta) = \langle \phi_j \mid V \mid \psi_i^{(+)} \rangle.$$  \hfill (2.64)

We can use the wave matrix to write this as

$$T(k', k) = \langle \phi_j \mid V\omega^{(+)} \mid \phi_i^{(+)} \rangle,$$  \hfill (2.65)

so that the transition amplitude is simply the matrix element of the operator $V\omega^{(+)}$ between free-particle states. We then denote the transition operator by

$$T = V\omega^{(+)},$$  \hfill (2.66)

so that

$$T\phi_i = V\psi_i^{(+)}.$$  \hfill (2.67)

The transition amplitude is just the matrix element of this operator:

$$T(k', k) = \langle \phi_j \mid T \mid \phi_i \rangle.$$  \hfill (2.68)

One often discusses scattering processes in terms of the "matrix element for the process," meaning the matrix element of the transition operator between unperturbed states $\phi_i$. The complete matrix involving transitions between all possible initial and final states is the $T$ matrix. (Often the operator $T$ itself is called the $T$ matrix.) We emphasize that the $T$ matrix gives directly the scattering amplitude whose determination is the goal of scattering theory.
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The transition operator $T$ is, of course, closely related to the potential operator $V$. If Eqs. (2.54) and (2.55) for the wave matrix are introduced into the definition of $T$, we find that the transition operator satisfies the operator equations

$$ T = V + V \frac{1}{E - H_0 + i\epsilon} T $$

(2.69)

or

$$ T = V + V \frac{1}{E - H + i\epsilon} V. $$

(2.70)

Summary:

$$ T = V + VG^{(+)}T $$

$$ T\phi = V\psi $$

$$ \psi = \phi + G^{(+)}V\psi $$
Watson Scattering Formalism
(Based on Watson, Phys. Rev. 89, 575 (1953) with updated notation)

2.1. Watson formalism

The multiple scattering optical potential formalism of Watson [Wa 53] provides a formal solution of the nonrelativistic many-body Schrödinger equation for the projectile–nucleus system in terms of a systematic expansion in elemental, many-body operators defined in ref. [Wa 53]. The many-body Schrödinger equation is

\[ \left( H_0 + H_p + H_A + \sum_{i=1}^{A} v_i \right) \psi = E \psi , \]  

(2.1)

where \( H_0 \) is the kinetic energy operator acting on the relative projectile–nucleus coordinates, \( H_p \) is the internal Hamiltonian of the projectile, \( H_A \) is the target nucleus Hamiltonian, \( v_i \) are the two-body projectile–constituent target nucleon interaction potentials (assumed to be the same in the nuclear medium as in free space, where no three-body or higher \( n \)-body forces are considered), \( A \) is the number of target nucleons, \( E \) is the total energy of the system in the center-of-momentum reference frame, and \( \psi \) is the total projectile–nucleus wave function. For energies in excess of the pion production threshold we must also consider the internal degrees of freedom of hadronic projectiles as well as those of the constituent target nucleons. The dynamics associated with these extra degrees of freedom are included in \( H_p \) and \( H_A \). The formal Lippmann–Schwinger [Li 50] solution of eq. (2.1) is

\[ \psi = \phi + \frac{1}{E - H_0 - H_p - H_A + i \delta} \sum_{i} v_i \psi , \]  

(2.2)

where \( \phi \) is the solution of the homogeneous equation

\[ (E - H_0 - H_p - H_A) \phi = 0 , \]  

(2.3)

and the \(+i\delta\) insures the proper boundary conditions for scattering.
Watson Scattering Formalism

In order to evaluate the projectile–nucleus scattering amplitude we will formulate the solution of eq. (2.2) in terms of the full projectile–nucleus $t$-matrix, defined by

$$ T\phi = \sum_i v_i \Psi . $$

The formal solution to the scattering problem is therefore

$$ T = \sum_i v_i + \sum_i v_i G(E)T , $$

where

$$ G(E) = (E - H_0 - H_p - H_A + i\delta)^{-1} $$

is the $(A+1)$-body propagator. Individually, each term in the expansion of $T$ in eq. (2.5) is very large for hadronic projectiles (infinite for NN potentials with hard repulsive cores) and the series converges poorly.

For projectile–nucleus scattering we will only be interested in matrix elements of $T$ between initial and final projectile–nucleus wave functions, representing physical states of the system. The symmetry properties of $(\Sigma_i v_i)$ [Fe 71] result in intermediate states in the second term of eq. (2.5) which span only the physical, antisymmetric states of the nucleus and the physical states of the projectile. Likewise, from eq. (2.4), only the projections of $\Psi$ and $\phi$ onto physical states of the system need be retained. Thus we consider $\Psi$ and $\phi$ from here on to be expanded in terms of all antisymmetric states of the target and all physical states of the projectile. Note that these states do not form complete sets [Ke 59].

Antisymmetrization between the projectile and target nucleon labels (in the case of nucleon projectiles) is, for the moment, neglected. The total wave function $\Psi$ is therefore expanded according to

$$ \Psi = \sum_{i,j} \psi_{i,j} \xi_i \Phi_j , $$

where

$$ H_p \xi_i = \epsilon_{p,i} \xi_i , $$

$$ H_A \Phi_j = \epsilon_{A,j} \Phi_j . $$
Watson Scattering Formalism

In eqs. (2.7)–(2.9) $\xi_i$ is a physical eigenstate of $H_p$ and $\Phi_i$ is an antisymmetrized eigenstate of $H_A$. $\psi_{ij}$ is the wave function describing the relative projectile–target motion. The homogeneous solution may be similarly expanded as

$$\phi = \sum_{ij} \varphi_{ij} \xi_i \Phi_j,$$

(2.10)

where $\varphi$ describes free propagation in the relative projectile–target nucleus coordinates.

It is convenient to introduce the projection operators [Fe 58] $\mathcal{A}$, $\mathcal{P}$, and $\mathcal{Q}$. The first is defined, using Dirac notation, by

$$\mathcal{A} = \sum_{ijk} |\varphi_k\rangle \langle \varphi_k| \xi_i \rangle \langle \xi_i| \Phi_j \rangle \langle \Phi_j|,$$

(2.11)

where the sums $i$ and $j$ include all physical projectile and target states, and the sum over $k$ includes the complete set of plane wave states for the projectile–target partition. For example, $\Sigma_k |\varphi_k\rangle \langle \varphi_k|$ becomes $(2\pi)^{-3} \int d^3k |\mathbf{k}\rangle \langle \mathbf{k}|$ for representation in momentum space. The elastic channel projection operator, $\mathcal{P}$, is

$$\mathcal{P} = \sum_k |\varphi_k\rangle \langle \varphi_k| \xi_0 \rangle \langle \xi_0| \Phi_0 \rangle \langle \Phi_0|,$$

(2.12)

where subscript $0$ denotes the ground states of the projectile and target nucleus. The nonelastic projection operator $\mathcal{Q}$ is simply

$$\mathcal{Q} = \mathcal{A} - \mathcal{P}.$$  

(2.13)

As usual $\mathcal{A}^2 = \mathcal{A}$, $\mathcal{P}^2 = \mathcal{P}$, $\mathcal{Q}^2 = \mathcal{Q}$, $\mathcal{A}\mathcal{P} = \mathcal{P}$, $\mathcal{A}\mathcal{Q} = \mathcal{Q}$, $\mathcal{P}\mathcal{Q} = 0$, and $[G, \mathcal{A}] = [G, \mathcal{P}] = [G, \mathcal{Q}] = 0$. Using these projection operators, eq. (2.5) for $T$ for physical projectile–nucleus scattering becomes

$$T = \sum_i v_i + \sum_i v_i G(E) \mathcal{A} T \quad \text{Intractable, non-perturbative…}$$

(2.14)
This auxiliary $t$-matrix is readily approximated in terms of the free N+N scattering amplitude (Chew's impulse approximation.) Perturbative expansions can be constructed in which many-body corrections are included.

The $t$-matrix introduced in the Watson approach is

$$t_i^W = v_i + v_i G(E) \varphi t_i^W,$$

which collects an infinite set of ladder diagrams involving the projectile's successive interactions with the same (ith) target nucleon. We refer to this as a quasi-two-body $t$-matrix. Elastic scattering matrix elements of $t_i^W$ are expected to be finite and of the order of the free two-body scattering amplitude. Formally solving eq. (2.15) for $v_i$, substituting into eq. (2.14), and defining

$$\sum_{i=1}^A T_i = T,$$ (2.16)

we obtain

$$T_i = t_i^W + t_i^W G(E) \mathcal{A} T - t_i^W G(E) \varphi T_i,$$ (2.17)

and therefore

$$T = \sum_i t_i^W + \sum_i t_i^W G(E) \mathcal{A} T - \frac{1}{A} \sum_i t_i^W G(E) \varphi T.$$ (2.18)

Projecting onto the elastic channel in eq. (2.18) yields

$$\mathcal{P} T \mathcal{P} = \sum_i \mathcal{P} t_i^W \mathcal{P} + \sum_i \mathcal{P} t_i^W G(E) (\mathcal{P} + \mathcal{L}) T \mathcal{P} - \frac{1}{A} \sum_i \mathcal{P} t_i^W G(E) \varphi T \mathcal{P},$$ (2.19)
Watson Scattering Formalism

which also requires that we evaluate \( \mathcal{P} \mathcal{T} \mathcal{P} \), given by

\[
\mathcal{P} \mathcal{T} \mathcal{P} = \sum_i \mathcal{P} t_i^w \mathcal{P} + \sum_i \mathcal{P} t_i^w G(E)(\mathcal{P} + \mathcal{2}) \mathcal{T} \mathcal{P} - \frac{1}{A} \sum_i \mathcal{P} t_i^w G(E) \mathcal{P} \mathcal{T} \mathcal{P} .
\]  

(2.20)

Formally solving eq. (2.20) for \( \mathcal{P} \mathcal{T} \mathcal{P} \) and substituting the result into eq. (2.19) yields an equation of the form

\[
\mathcal{P} \mathcal{T} \mathcal{P} = \mathcal{P} U_w^\mathcal{P}[1 + G(E) \mathcal{P} \mathcal{T} \mathcal{P}] ,
\]

(see slides 22-24

for details)

where \( \mathcal{P} U_w^\mathcal{P} \), given by

\[
\mathcal{P} U_w^\mathcal{P} = \sum_i \mathcal{P} t_i^w \mathcal{P} + \frac{A-1}{A} \sum_i \mathcal{P} t_i^w \mathcal{2} \left( G(E)^{-1} - \frac{A-1}{A} \sum_k \mathcal{2} t_k^w \mathcal{2} \right)^{-1} \sum_i \mathcal{2} t_i^w \mathcal{P} .
\]

(2.22)

is the desired projectile–nucleus optical potential for elastic scattering in the Watson formalism. This optical potential can be rewritten in an equivalent form,

\[
\mathcal{P} U_w^\mathcal{P} = \sum_i \mathcal{P} t_i^w \mathcal{P} + A(A-1) \left( \frac{1}{A(A-1)} \sum_{i \neq i} \mathcal{P} t_i^w \frac{A}{\hat{a}_w} t_i^w \mathcal{P} - \frac{1}{A} \sum_i \mathcal{P} t_i^w \mathcal{P} \right) \left( \frac{1}{\hat{a}_w} A \sum_i \mathcal{P} t_i^w \mathcal{P} \right),
\]

(2.23)

where

\[
\hat{a}_w = G(E)^{-1} - \frac{A-1}{A} \sum_k \mathcal{2} t_k^w \mathcal{2} .
\]

(2.24)

Solve Sch.Eq. for elastic wave function and scattering amplitude.

The projectile–nucleus elastic scattering amplitude may also be obtained by solving a one-body Schrödinger equation for the elastic channel using the optical potential \( \mathcal{P} U_w^\mathcal{P} \), where from eq. (2.21) the wave equation to be evaluated is

\[
\mathcal{P} [H_0 + \epsilon_{\nu,0} + \epsilon_{\nu,0} + U^w] \mathcal{P} \Psi = E \mathcal{P} \Psi .
\]

(2.25)

Solution of eq. (2.25), together with the appropriate asymptotic boundary conditions, results in the projectile–nucleus scattering amplitude.
Watson Scattering Formalism

Details from preceding slides

\[ T = \sum_{i=1}^{A} v_i + \sum_{i=1}^{A} v_i G^{(+)}(E) A T, \text{ just write } G^{(+)} \text{ for brevity.} \]

\[ t_i^w = v_i + v_i G^{(+)} A t_i^w, \text{ solve for } v_i \]

\[ v_i = t_i^w \frac{1}{1 + G^{(+)} A t_i^w} = \frac{1}{1 + t_i^w G^{(+)} A} t_i^w \]

\[ T = \sum_{i=1}^{A} \frac{1}{1 + t_i^w G^{(+)} A} t_i^w + \sum_{i=1}^{A} \frac{1}{1 + t_i^w G^{(+)} A} t_i^w G^{(+)} A T \]

define \( T = \sum_{i=1}^{A} T_i \), in terms of the transition amplitude for target nucleon \( i \)

\[ T_i = \frac{1}{1 + t_i^w G^{(+)} A} t_i^w + \frac{1}{1 + t_i^w G^{(+)} A} t_i^w G^{(+)} A T, \text{ left operate with } 1 + t_i^w G^{(+)} A \]

\[ (1 + t_i^w G^{(+)} A) T_i = t_i^w + t_i^w G^{(+)} A T \]

\[ T_i = t_i^w + t_i^w G^{(+)} A T - t_i^w G^{(+)} A T_i \]

\[ T = \sum_{i=1}^{A} t_i^w + \sum_{i=1}^{A} t_i^w G^{(+)} A T - \sum_{i=1}^{A} t_i^w G^{(+)} A T_i \]

For antisymmetric nuclear states all \( T_i \) are equal. Hence \( T_i = \frac{1}{A} \sum_{i=1}^{A} T_i = \frac{1}{A} T \)

\[ T = \sum_{i=1}^{A} t_i^w + \sum_{i=1}^{A} t_i^w G^{(+)} A T - \frac{1}{A} \sum_{i=1}^{A} t_i^w G^{(+)} A T \]. Project out the elastic scattering channel

\[ PTP = \sum_{i=1}^{A} P_i^w P + \sum_{i=1}^{A} P_i^w G^{(+)} (P + Q) T P - \frac{1}{A} \sum_{i=1}^{A} P_i^w G^{(+)} A T P, \text{ and we also need } QTP \]

\[ QTP = \sum_{i=1}^{A} Q_i^w P + \sum_{i=1}^{A} Q_i^w G^{(+)} (P + Q) T P - \frac{1}{A} \sum_{i=1}^{A} Q_i^w G^{(+)} A T P \]
\[ PTP = \sum_{i=1}^{A} Pt_i^w \mathcal{P} + \sum_{i=1}^{A} Pt_i^w G^{(+)}(P + Q)\mathcal{P} - \frac{1}{A} \sum_{i=1}^{A} Pt_i^w G^{(+)} QTP \]

\[ = \sum_{i=1}^{A} Pt_i^w \mathcal{P} + \sum_{i=1}^{A} Pt_i^w G^{(+)} PTP + \frac{A-1}{A} \sum_{i=1}^{A} Pt_i^w G^{(+)} QTP \]

Solve the following for \( QTP \):

\[ QTP = \sum_{i=1}^{A} Qt_i^w \mathcal{P} + \sum_{i=1}^{A} Qt_i^w G^{(+)}(P + Q)\mathcal{P} - \frac{1}{A} \sum_{i=1}^{A} Qt_i^w G^{(+)} QTP \]

\[ = \sum_{i=1}^{A} Qt_i^w \mathcal{P} + \sum_{i=1}^{A} Qt_i^w G^{(+)} PTP + \frac{A-1}{A} \sum_{i=1}^{A} Qt_i^w G^{(+)} QTP \]

\[ \left[1 - \frac{A-1}{A} \sum_{k=1}^{A} Q_k^w QG^{(+)}\right] QTP = \sum_{i=1}^{A} Qt_i^w \mathcal{P}[1 + G^{(+)} PTP] \]

\[ QTP = \left[1 - \frac{A-1}{A} \sum_{k=1}^{A} Q_k^w QG^{(+)}\right]^{-1} \sum_{i=1}^{A} Qt_i^w \mathcal{P}[1 + G^{(+)} PTP] \], substitute into the above eqn.

\[ PTP = \sum_{i=1}^{A} Pt_i^w \mathcal{P}[1 + G^{(+)} PTP] + \frac{A-1}{A} \sum_{i=1}^{A} Pt_i^w QG^{(+) \left\{ \left[1 - \frac{A-1}{A} \sum_{k=1}^{A} Q_k^w QG^{(+)}\right]^{-1} \sum_{j=1}^{A} Qt_j^w \mathcal{P}[1 + G^{(+)} PTP] \right\} \]

\[ = \left\{\sum_{i=1}^{A} Pt_i^w \mathcal{P} + \frac{A-1}{A} \sum_{i=1}^{A} Pt_i^w QG^{(+) \left\{ \left[1 - \frac{A-1}{A} \sum_{k=1}^{A} Q_k^w QG^{(+)}\right]^{-1} \sum_{j=1}^{A} Qt_j^w \mathcal{P}\right\} \right\} \mathcal{P}[1 + G^{(+)} PTP] \]

\[ \mathcal{P}U^w \mathcal{P} = \sum_{i=1}^{A} Pt_i^w \mathcal{P} + \frac{A-1}{A} \sum_{i=1}^{A} Pt_i^w Q \left[G^{(+)}^{-1} - \frac{A-1}{A} \sum_{k=1}^{A} Q_k^w Q\right]^{-1} \sum_{j=1}^{A} Qt_j^w \mathcal{P} \], optical potential;

where the operator identity* \[ C \frac{1}{1 - BC} = \frac{1}{C^{-1} - B} \] was used for the propagator.

*Starting with \((1 - BC)\frac{1}{1 - BC} = 1\), then \((C^{-1} - B)C \frac{1}{1 - BC} = 1\) and we get \[ C \frac{1}{1 - BC} = \frac{1}{C^{-1} - B} \]
Watson Scattering Formalism

Details continued,

Add & subtract the elastic channel

\[
P_U^w P = \sum_{i=1}^{A} P t_i^w P + \frac{A-1}{A} \left[ \sum_{i,j} P t_i^w Q \frac{1}{\tilde{\alpha}_w} Q t_j^w P + \sum_{i,j} P t_i^w P \frac{1}{\tilde{\alpha}_w} P t_j^w P - \sum_{i,j} P t_i^w P \frac{1}{\tilde{\alpha}_w} P t_j^w P \right],
\]

where \( \tilde{\alpha}_w = G^{(+)}^{-1} - \frac{A-1}{A} \sum_{k=1}^{A} Q t_k^w Q \)

\[
P_U^w P = \sum_{i=1}^{A} P t_i^w P + \frac{A-1}{A} \left[ \sum_{i,j} P t_i^w \frac{A}{\tilde{\alpha}_w} t_j^w P - \sum_{i,j} P t_i^w P \frac{1}{\tilde{\alpha}_w} P t_j^w P \right]
\]

\[
= \sum_{i=1}^{A} P t_i^w P + A(A-1) \left[ \frac{1}{A^2} \sum_{i,j} P t_i^w \frac{A}{\tilde{\alpha}_w} t_j^w P - \frac{1}{A^2} \sum_{i,j} P t_i^w P \frac{1}{\tilde{\alpha}_w} P t_j^w P \right]
\]

\[
P_U^w P = \sum_{i=1}^{A} P t_i^w P + A(A-1) \left[ \frac{1}{A(A-1)} \sum_{i \neq j} P t_i^w \frac{A}{\tilde{\alpha}_w} t_j^w P - \frac{1}{A^2} \sum_{i,j} P t_i^w P \frac{1}{\tilde{\alpha}_w} P t_j^w P \right] (\text{see FGH Appendix})
\]

Therefore the non-perturbative expansion

\[
T = \sum_i v_i + \sum_i v_i G^{(+)}(E) A T
\]

has been reorganized in terms of two, perturbative expansions. The first involves many-body corrections to the quasi-free two-nucleon t-matrix; the second involves corrections to the optical potential involving correlations in the nuclear wave function.
KMT Scattering Formalism

(Based on KMT, Ann. Phys. 8, 551 (1959) with updated notation)

2.2. KMT formalism

The reformulation of the Watson multiple scattering theory by Kerman, McManus and Thaler [Ke 59] was motivated by the need to have an optical potential constructed from quasi-two-body operators which are better approximated by the free, two-body scattering $t$-matrices (in the spirit of the IA). Therefore in the KMT approach the quasi-two-body operators used are defined by

$$t_i^K = v_i + v_i G(E) A_i^K,$$  \hspace{1cm} (2.26)

where, unlike the Watson operator $t_i^W$, the full $(A + 1)$-body spectrum projected by $A$ is included in intermediate states. Solving for $v_i$, and substituting the result into eq. (2.14), we obtain

$$T = \sum_i t_i^K + \sum_i t_i^K G(E) A T - \frac{1}{A} \sum_i t_i^K G(E) A T,$$  \hspace{1cm} (2.27)

where the definition in eq. (2.16) has again been used. In a manner analogous to that leading from eq. (2.18) to eqs. (2.21) and (2.22) we obtain

$$\mathcal{P}T\mathcal{P} = \left[\sum_i \mathcal{P}t_i^K P + \frac{A-1}{A} \sum_i \mathcal{P}t_i^K \mathcal{P} \left( G(E)^{-1} - \frac{A-1}{A} \sum_k \mathcal{A} t_k^K \mathcal{A} \right)^{-1} \sum_i \mathcal{A} t_i^K \mathcal{A} \right]$$

$$\times \left(1 + \frac{A-1}{A} G(E) \mathcal{P}T\mathcal{P}\right).$$  \hspace{1cm} (2.28)

However, this is not in the form of the Lippmann–Schwinger (LS) equation, due to the $(A-1)/A$ factor multiplying the $G\mathcal{P}T\mathcal{P}$ term. To get eq. (2.28) into LS form, KMT introduced an alternate $t$-matrix, $T'$, where

$$T' = \frac{A-1}{A} T.$$  \hspace{1cm} (2.29)

Using this $T'$, eq. (2.28) becomes

$$\mathcal{P}T'\mathcal{P} = \mathcal{P}U^K\mathcal{P} + \mathcal{P}U^K\mathcal{P} G(E) \mathcal{P}T'\mathcal{P},$$  \hspace{1cm} (2.30)
KMT Scattering Formalism

where the KMT optical potential is given by

\[
P U^K_P = \frac{A - 1}{A} \sum_i P_t^K_i P + \left( \frac{A - 1}{A} \right)^2 \sum_i P_t^K_i 2 \left( G(E)^{-1} - \frac{A - 1}{A} \sum_k 2 t_k^K 2 \right)^{-1} \sum_j 2 t_j^K P. \quad (2.31)
\]

In the form analogous to eqs. (2.23) and (2.24) this becomes,

\[
P U^K_P = \frac{A - 1}{A} \sum_i P_t^K_i P
\]

\[
+ (A - 1)^2 \left( \frac{1}{A(A - 1)} \sum_{i \neq j} P_t^K_i \frac{A}{\tilde{\alpha}_K} t_j^K P - \frac{1}{A} \sum_i P_t^K_i P \frac{1}{\tilde{\alpha}_K} \frac{1}{A} \sum_j P_t^K_j P \right), \quad (2.32)
\]

where

\[
\tilde{\alpha}_K = G(E)^{-1} - \frac{A - 1}{A} \sum_k 2 t_k^K 2. \quad (2.33)
\]

Thus the KMT optical potential can be obtained from the Watson potential by substituting \( t_i^K \) for \( t_i^W \) and multiplying by \( (A - 1)/A \).

Offsetting the advantages of the KMT approach are the following: (1) Including the Coulomb interaction is cumbersome due to the \( (A - 1)/A \) scaling factors [Ra 80]. (2) Correctly accounting for the isospin dependence of the two-body operators is again complicated by the \( (A - 1)/A \) factors [Ra 83b].
KMT Scattering Formalism

The Watson and KMT optical potentials are easily related using the expressions for the full $t$-matrix

$$\mathcal{P}T\mathcal{P} = \mathcal{P}U^w\mathcal{P} + \mathcal{P}U^w\mathcal{P}G(E)\mathcal{P}T\mathcal{P},$$  \hspace{1cm} (2.34)

$$\mathcal{P}T\mathcal{P} = \frac{A}{A-1} \mathcal{P}U^K\mathcal{P} + \mathcal{P}U^K\mathcal{P}G(E)\mathcal{P}T\mathcal{P}.$$  \hspace{1cm} (2.35)

Eliminating $\mathcal{P}T\mathcal{P}$ in eqs. (2.34) and (2.35) results in

$$\mathcal{P}U^w\mathcal{P} = \frac{A}{A-1} \mathcal{P}U^K\mathcal{P} - \frac{1}{A-1} \mathcal{P}U^K\mathcal{P}G(E)\mathcal{P}U^w\mathcal{P},$$  \hspace{1cm} (2.36)

or

$$\mathcal{P}U^K\mathcal{P} = \frac{A-1}{A} \mathcal{P}U^w\mathcal{P} + \frac{1}{A} \mathcal{P}U^w\mathcal{P}G(E)\mathcal{P}U^K\mathcal{P}.$$  \hspace{1cm} (2.37)

It is also useful to relate the Watson and KMT quasi-two-body operators to each other and to the free two-body $t$-matrix, $t_i$, given by

$$t_i = v_i + v_ig(\varepsilon)t_i,$$  \hspace{1cm} (2.38)

where

$$g(\varepsilon) = (\varepsilon - h_0 - H_P - H_N + i\delta)^{-1}. \hspace{1cm} (2.39)$$
KMT Scattering Formalism

The free two-body energy is \( \varepsilon \), \( h_0 \) is the kinetic energy operator acting on the relative two-body coordinate, and the internal Hamiltonians of the projectile and target nucleon constituent are \( H_p \) and \( H_N \), respectively. These relations are

\[
\begin{align*}
    t_i^W &= t_i^K - t_i^K G(E) \mathcal{D} t_i^W, \\
    t_i^K &= t_i^W + t_i^W G(E) \mathcal{D} t_i^K, \\
    t_i^W &= t_i + t_i[G(E) \mathcal{D} - g(\varepsilon)] t_i^W, \\
    t_i^K &= t_i + t_i[G(E) \mathcal{A} - g(\varepsilon)] t_i^K.
\end{align*}
\]  

(2.40) (2.41) (2.42) (2.43)

Expansion of the effective, two-body interaction in terms of the free-space \( N+N \) scattering amplitude.
APPENDIX A: Derivation of the Second Order Optical Potential

A careful treatment of permutation-symmetry is required when dealing with scattering processes that occur in a many-nucleon system. In the following we shall strictly observe the generalized Pauli principle by defining physical quantities only in the antisymmetric subspace of the many-nucleon space. Because of the high energy of the projectile, the antisymmetry requirement for the projectile, in case it is a nucleon, will be neglected.

Let \( v_i \) stand for the primary interaction between the projectile and the \( i \)th nucleon of the target. The \( T \)-matrix is defined by the integral equation it satisfies

\[
T = \sum_{i=1}^{N} v_i \mathcal{A} + \sum_{i=1}^{N} v_i \frac{\mathcal{A}}{T} \tag{A.1}
\]

where \( \mathcal{A} \) is a projection operator onto the antisymmetric subspace of the target-nucleons and \( \alpha \) is the free hamiltonian (\( H_N \) is the nuclear hamiltonian)

\[
\alpha = E^{(+)} - K - H_N \tag{A.2}
\]

Matrix elements of \( T \), defined by Eq. (A.1), between antisymmetric states are the usual ones, while matrix elements of \( T \) between antisymmetric states and states of other types of symmetry vanish. This follows because of the symmetry of the interaction \( \sum v_i \) and because the homogeneous equation corresponding to Eq. (A.1) has no nontrivial solutions. This is true also for matrix elements in states of other than antisymmetric type of symmetry.
Feshbach, Gal and Hufner Scattering Formalism

Next, we define the effective interaction $\tau$ by

$$\tau = \frac{1}{N} \sum v_i \mathcal{A} + \frac{1}{N} \sum \frac{v_i}{\alpha} \tau. \quad (A.3)$$

The permutation properties of $\tau$ are the same as for $T$, according to the discussion above. Equation (A.3) yields the following relation for the combination $1/N \sum v_i \mathcal{A}$:

$$\frac{1}{N} \sum v_i \mathcal{A} = \left( 1 - \frac{1}{N} \sum \frac{v_i}{\alpha} \right) \tau.$$ 

Substituting in (A.1) yields

$$T = N \left( 1 - \frac{1}{N} \sum \frac{v_i}{\alpha} \right) \tau + N \left( 1 - \frac{1}{N} \sum \frac{v_i}{\alpha} \right) \tau \frac{1}{\alpha} T$$

$$= N\tau - \sum \frac{v_i}{\alpha} \tau + N\tau \frac{1}{\alpha} T - \sum \frac{v_i}{\alpha} \tau \frac{1}{\alpha} T$$

$$= N\tau + N\tau \frac{1}{\alpha} T - \tau \frac{1}{\alpha} \left( \sum v_i \mathcal{A} + \sum \frac{v_i}{\alpha} \frac{1}{\alpha} T \right)$$

$$= N\tau + N\tau \frac{1}{\alpha} T - \tau \frac{1}{\alpha} T$$

$$= N\tau + (N - 1) \tau \frac{1}{\alpha} T,$$

where use was made of the relation $\tau(1/\alpha) \sum v_i \mathcal{A} = \sum v_i (\mathcal{A}/\alpha) \tau$. Introducing Eq. (2.2)

$$T = \frac{N}{N - 1} T',$$
Feshbach, Gal and Hufner Scattering Formalism

we obtain

\[ T' = (N - 1) \tau + (N - 1) \tau \frac{1}{\alpha} T' \]  

(A.4)  KMTT' - matrix

which now may be compared with (2.4).

For comparison with the free nucleon projectile scattering, we define the two quantities

\[ \tau_i = v_i A^i + v_i \frac{A}{\alpha} \tau \]  

(A.5)  (many-body propagator)

and

\[ t_i = v_i + v_i \frac{1}{\alpha} t_i \]  

(A.6)

Note that

\[ \tau = \frac{1}{N} \sum \tau_i \]  

(A.7)  \[ \tau A \tau_i = \tau_i \]  

Except for the \( \alpha \)-denominator, \( t_i \) is the appropriate scattering operator. For energetic projectiles this is a small correction. We would like to find the relation between the \( \tau_i \) and \( t_i \). This is done in analogy with the derivation of Eq. (A.4):

\[ \tau_i = t_i \left( 1 - \frac{1}{\alpha} v_i \right) A^i + t_i \left( 1 - \frac{1}{\alpha} v_i \right) \frac{A}{\alpha} \tau \]

Solve (A.6) for \( v_i \), sub. into (A.5)

\[ = t_i A^i - t_i \frac{1}{\alpha} v_i A^i + t_i \frac{A}{\alpha} \tau - t_i \frac{1}{\alpha} (\tau_i - v_i A) \]

\[ = t_i A^i + t_i \frac{A}{\alpha} \tau - t_i \frac{1}{\alpha} \tau_i \]

\[ = t_i A^i + t_i \frac{1}{\alpha} (\tau - \tau_i) , \]

\( A \) is redundant with definition of \( \tau \) in (A.3)

where \( A \) in front of the \( \tau \) operator was omitted due to the symmetry property of \( \tau \).
where \( \mathcal{A} \) in front of the \( \tau \) operator was omitted due to the symmetry property of \( \tau \).

As a result,\
\[
\tau = \frac{1}{N} \sum_i \langle i | \mathcal{A} t_i \rangle = \frac{1}{N} \sum_i t_i \mathcal{A} + \frac{1}{N} \sum_i t_i \frac{1}{\alpha} (\tau - \tau_i)
\]
\[
= \frac{1}{N} \sum t_i \mathcal{A} + \frac{1}{N} \sum t_i \frac{\mathcal{A} - 1}{\alpha} \tau_i,
\]
(A.8)

where we have used Eq. (A.7). We turn now to the optical potential defined by Eq. (2.3) in terms of the operator \( \tau \). The first-order potential is given by the nuclear-ground-state matrix element of \( (N - 1)\tau \). We define an effective hamiltonian by
\[
\tilde{\alpha} = E^{(+)} - K - H_N - (N - 1)\tau,
\]
(A.9)

the relation of which to the \( \alpha \) defined in (A.2) is
\[
\frac{1}{\alpha} = \frac{1}{\tilde{\alpha}} - \frac{1}{\tilde{\alpha}} (N - 1) \tau \frac{1}{\tilde{\alpha}}.
\]
(A.10)

Since \( \tau \) vanishes outside the antisymmetric subspace and \( \alpha \) and \( \tilde{\alpha} \) cannot change the symmetry type, application of \( \mathcal{A} - 1 \) to both sides of (A.10) gives
\[
\frac{\mathcal{A} - 1}{\alpha} = \frac{\mathcal{A} - 1}{\tilde{\alpha}}.
\]
(A.11)

Hence,
\[
\langle \tau \rangle_{00} = \mathcal{P} \tau \mathcal{P}
\]
\[
(N - 1)\langle \tau \rangle_{00} = \frac{N - 1}{N} \sum_i t_i \langle i | \mathcal{A} \rangle_{00} + \frac{N - 1}{N} \sum_i t_i \frac{\mathcal{A} - 1}{\tilde{\alpha}} \tau_i \langle i | \mathcal{A} \rangle_{00}.
\]
(A.12)

We approximate \( \tau_i \) by \( t_i \) in the quadratic term (this affects third order and on):
\[
(N - 1)\langle \tau \rangle_{00} \approx \frac{N - 1}{N} \sum_i t_i \langle i | \mathcal{A} \rangle_{00} + \frac{N - 1}{N} \sum_i t_i \frac{\mathcal{A} - 1}{\tilde{\alpha}} \tau_i \langle i | \mathcal{A} \rangle_{00}
\]
\[
= \frac{N - 1}{N} \sum_i t_i \langle i | \mathcal{A} \rangle_{00} + \frac{N - 1}{N} \sum_i \frac{1}{\tilde{\alpha}} \sum_j t_j \langle i | \mathcal{A} \rangle_{00}
\]
\[
- \frac{(N - 1)}{N} \sum_i \frac{1}{\tilde{\alpha}} t_i \langle i | \mathcal{A} \rangle_{00}.
\]
(A.13)
The second order of the optical potential Eq. (2.3) is

\[(N - 1)^2 \sum_{p \neq 0} \langle \tau \rangle_{0p} \frac{1}{\langle \beta | \tilde{\alpha} | \beta \rangle} \langle \tau \rangle_{0\beta}.\]

Consistently with the previous replacement of \( \tau \) by \( 1/N \sum t_i \tilde{\alpha} \) in quadratic terms, we repeat this replacement also here. The operator \( \tilde{\alpha} \) both here and in (A.13) is taken to be diagonal with the diagonal value, \( \tilde{\alpha} \), independent of the label \( \beta \). Then, by closure

\[(N - 1)^2 \left[ \langle \tau \frac{1}{\tilde{\alpha}} \tau \rangle_{00} - \langle \tau \rangle_{00} \frac{1}{\tilde{\alpha}} \langle \tau \rangle_{00} \right] \]

\[\simeq (N - 1)^2 \left[ \frac{1}{N^2} \left\langle \sum t_i \frac{1}{\tilde{\alpha}} \sum t_j \right\rangle_{00} - \frac{1}{N^2} \left\langle \sum t_i \right\rangle_{00} \frac{1}{\tilde{\alpha}} \left\langle \sum t_j \right\rangle_{00}. \] (A.14)
Feshbach, Gal and Hufner Scattering Formalism

Equations (A.13) and (A.14) are combined to give for $V^{\text{OPT}}$

\[
V^{\text{OPT}} = \frac{N - 1}{N} \langle \sum_i t_i \rangle_{00} + \frac{N - 1}{N^2} \langle \sum_i t_i \sum_j t_j \rangle_{00} - \frac{N - 1}{N} \langle \sum_i \frac{1}{\alpha} t_i \rangle_{00}
\]

\[
+ \frac{(N - 1)^2}{N^2} \langle \sum_i t_i \frac{1}{\alpha} \sum_j t_j \rangle_{00} - \frac{(N - 1)^2}{N^2} \langle \sum_i t_i \frac{1}{\alpha} \sum_j t_j \rangle_{00}
\]

\[
= \frac{N - 1}{N} \langle \sum_i t_i \rangle_{00} + \frac{N - 1}{N} \langle \sum_i t_i \sum_j t_j \rangle_{00} - \frac{N - 1}{N} \langle \sum_i \frac{1}{\alpha} t_i \rangle_{00}
\]

\[
- \frac{(N - 1)^2}{N^2} \langle \sum_i t_i \rangle_{00} \frac{1}{\alpha} \langle \sum_j t_j \rangle_{00}
\]

\[
= \frac{N - 1}{N} \langle \sum_i t_i \rangle_{00} + \frac{N - 1}{N} \langle \sum_i t_i \sum_{j \neq i} t_j \rangle_{00}
\]

\[
- \frac{(N - 1)^2}{N^2} \langle \sum_i t_i \rangle_{00} \frac{1}{\alpha} \langle \sum_j t_j \rangle_{00}.
\]

Finally

\[
V^{\text{OPT}} = (N - 1) \left( \frac{1}{N} \sum_i t_i \right)_{00}
\]

\[
+ (N - 1)^2 \left[ \frac{1}{N(N - 1)} \sum_{i \neq j} t_i \frac{1}{\alpha} t_j \right]_{00} - \langle \frac{1}{N} \sum t_i \rangle_{00} \frac{1}{\alpha} \langle \frac{1}{N} \sum t_j \rangle_{00}.
\]

(A.15)
Feshbach, Gal and Hufner Scattering Formalism

\[ V_{\text{OPT}} = (N - 1) \left\langle \frac{1}{N} \sum t_i \right\rangle_{00} + (N - 1)^2 \left\{ \frac{1}{N(N-1)} \sum_{i \neq j} \left\langle t_i \frac{1}{\bar{\alpha}} t_j \right\rangle_{00} - \frac{1}{N} \sum t_i \frac{1}{\bar{\alpha}} \left\langle \frac{1}{N} \sum t_j \right\rangle_{00} \right\} \]  

(A.15)

Average over particle pairs in nuclear ground-state; includes correlations

Product of single particle, ground state densities; uncorrelated

Proportional to true, two-particle correlations in the nuclear ground-state, weighted with the effective projectile + nucleon interaction squared.
Brueckner applied the multiple scattering theory of Watson to infinite nuclear matter (INM), but with a few changes. The goal of the INM problem is to calculate from the bare N+N interaction the binding energy and saturation density of INM as estimated by the semi-empirical mass formula (about 16 MeV/A) and the densities in the interiors of large nuclei (about 0.17 nucleons/fm$^3$). The main difference for INM from the scattering solution involves the structure of the effective two-body operators. For scattering the idea was to formulate the many-body problem in terms of effective two-body operators which can be well approximated with the free-space scattering amplitude. For INM there is no “free space” and we should build the following into the lowest-order effective interaction operator:

1) Pauli exclusion (interacting pairs of particles may only jump in to, and out of, unoccupied states above the Fermi surface), and
2) the average nucleon potential energy in INM in the propagator.

Also, in Brueckner’s theory two-nucleon correlations, discussed above, were neglected.
Brueckner effective interaction

For INM there are no outgoing spherical waves; the uniform, isotropic symmetry requires a nucleon's asymptotic form to be a plane wave

$$\phi_k = e^{i k \cdot \vec{r}}$$

and the energy to be $$E(k) = \frac{\hbar^2 k^2}{2m} + U(k)$$, where $$m$$ is the nucleon mass and $$U(k)$$ is the potential energy.

Also there is no "special" nucleon like the projectile in scattering. Operators therefore refer to any arbitrary pair of nucleons $$\alpha$$ with particle labels $$i, j$$. The two-body operator equation analogous to that for scattering is:

$$G_{\beta, \alpha} = v_\alpha + v_\alpha G_{\text{INM}} Q G_{\beta, \alpha}$$ (Bethe-Goldstone equation)

where $$v_\alpha$$ is the N+N interaction, $$G_{\text{INM}}$$ is the propagator in INM, and $$Q$$ projects only unoccupied intermediate states above the Fermi surface.

For scattering the energy denominator was$$\left( E - H_0 - H_\rho - H_\Lambda + i \varepsilon \right)$$. For INM the energy denominator is

$$\left( E - \frac{\hbar^2 k_i^2}{2m} - \frac{\hbar^2 k_j^2}{2m} - U_i - U_j - \sum_k T_k - \sum_{k<k'} v_{kk'} \right) = E - H_0 - V_C$$ where $$H_0$$ sums all kinetic energy operators, $$V_C$$ is the potential energy of INM, and $$k, k'$$ sum over all nucleons other than $$i, j$$. Intermediate states only permit excitations of the $$i, j$$ nucleon pair; the INM remains in its ground state, i.e. we only consider the ground state matrix element$$\langle \Psi_{gs} | V_C | \Psi_{gs} \rangle$. 
Brueckner effective interaction

From the algebra leading to the first - order optical potential this INM potential energy is

\[ \langle \Psi_{gs} | V_C | \Psi_{gs} \rangle \approx \sum_{i<j} \langle ij | G_{\beta,ij} | ij \rangle_A, \]

the sum over all diagonal elements where subscript A imposes anti - symmetric states. The above g.s. matrix element of \( G_\beta \) is analogous to the first - order, elastic - channel matrix element of the optical potential. For INM there is no absorption and this term is real, and gives just the potential energy. The propagator for INM is therefore given by

\[ G_{INM} = \frac{1}{E - H_0 - \sum_{i<j} \langle ij | G_{\beta,ij} | ij \rangle_A}. \]

For an arbitrary \( N + N \) pair \((i, j)\)

\[ \langle ij | G_\beta | ij \rangle_A = \langle ij | \nu | ij \rangle_A + \sum_{m,n} \frac{\langle ij | \nu | mn \rangle_A Q \langle mn | G_\beta | ij \rangle_A}{E_i + E_j - \frac{\hbar^2 k_m^2}{2m} - \frac{\hbar^2 k_n^2}{2m} - \sum_{i'<j'} \langle i'j' | G_{\beta,ij} | i'j' \rangle_A}, \]

and

\[ Q \langle mn | A \rangle \text{ if both states} | m \rangle \text{ and} | n \rangle \text{ are above the Fermi surface, and zero otherwise.} \]

The average, pair - wise potential energy is given by \( \sum_{i<j} \langle ij | G_\beta | ij \rangle_A \) where

\[ \sum_{i<j} \langle ij | G_\beta | ij \rangle_A = \sum_{i<j} \langle ij | \nu | ij \rangle_A + \sum_{i<j,m,n} \frac{\langle ij | \nu | mn \rangle_A Q \langle mn | G_\beta | ij \rangle_A}{E_i + E_j - \frac{\hbar^2 k_m^2}{2m} - \frac{\hbar^2 k_n^2}{2m} - \sum_{i'<j'} \langle i'j' | G_{\beta,ij} | i'j' \rangle_A}. \]

This must be solved iteratively.
Relativistic multiple scattering & effective interactions


A semi-relativistic scattering model for p + A in Hamiltonian form:

$$\left[ \bar{\alpha} \cdot \bar{p} + \beta \left( m + \sum_{i=1}^{A} v_{pi} \right) + H_A \right] \Psi = E \Psi$$

where $\bar{\alpha}, \beta$ are the usual Dirac matrices for the projectile proton. The semi-relativistic propagator is

$$G = \left( p_\mu \gamma^\mu - m - \gamma^0 H_A + i \varepsilon \right)^{-1}$$

and the semi-relativistic many-body T-matrix is

$$T = \sum_{i=1}^{A} v_{pi} + \sum_{i=1}^{A} v_{pi} G A T$$

Projecting the elastic channel and following the above steps for the Watson optical potential gives

$$\mathcal{P} T \mathcal{P} = \mathcal{P} U^D \mathcal{P} + \mathcal{P} U^D \mathcal{P} G \mathcal{P} T \mathcal{P}$$

where $\mathcal{P} U^D \mathcal{P}$ is the Dirac p + A optical potential

$$\mathcal{P} U^D \mathcal{P} = \langle \Phi_{gs} \sum_i t_i | \Phi_{gs} \rangle + A(A-1) \left[ \frac{1}{A(A-1)} \sum_{i \neq j} \langle \Phi_{gs} | \bar{G} t_j | \Phi_{gs} \rangle \right]$$

$$- \frac{1}{A^2} \langle \Phi_{gs} \sum_i t_i | \Phi_{gs} \rangle \bar{G} \langle \Phi_{gs} \sum_j t_j | \Phi_{gs} \rangle$$

$$\bar{G} = \left( p_\mu \gamma^\mu - m - \gamma^0 \bar{E}_A + i \varepsilon \right)^{-1}$$

where $\bar{E}_A$ is an average, intermediate nuclear excitation energy, and

$$t_i = v_{pi} A + v_{pi} G Q t_i \xrightarrow{\text{impulseapprox}} v_{pi} + v_{pi} g t_i$$

where $t_i$ is given in S, P, V, A, T form (see Chpt. I).

The nuclear ground state here is constructed from anti-symmetric combinations (Slater determinant) of relativistic single-particle states

$$u_{nlj}(r) = \begin{pmatrix} \phi_{nlj}(r) \Psi_{ij}^\mu(\hat{r}) \\ i \lambda_{nlj}(r) \Psi_{ij}^\mu(\hat{r}) \end{pmatrix}$$

where $\Psi_{ij}^\mu(\hat{r})$ is the spin-angle function (see Chpt. I), $\phi$ and $\lambda$ are the positive-energy upper and lower components, and $\tilde{\ell} = \ell \pm 1$ for $j = \ell \pm \frac{1}{2}$. 
This concludes Chapter 3: Effective Interaction Theory